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Goldstone mode singularities in specific heats and non-ordering susceptibilities of isotropic systems

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Abstract. Goldstone mode singularities in non-ordering correlation functions of the $O(N)$-symmetric ϕ^4 theory are investigated in $4-\varepsilon$ dimensions. Functions constructed from non-invariant combinations of the transverse operator π^2 and the longitudinal operators σ^2 and σ generally exhibit severe infrared divergences on the coexistence curve. In low-order perturbative calculations of, for example, the specific heat, however, these divergences cancel. Such cancellations are shown to be a general feature of $O(N)$-symmetric correlation functions. Non-leading singularities remain, and lead to momentum dependence which can be expressed as a double power series in $|p|^{\varepsilon}$ and $|p|^{2-\varepsilon}$.

1. Introduction

It is well known that the ordering susceptibilities of isotropic systems diverge at the coexistence curve. If H is the external field coupled to the order parameter (e.g. a uniform magnetic field in the case of an isotropic ferromagnet) then in $d=4-\varepsilon$ dimensions, the transverse and longitudinal susceptiblities behave as H^{-1} and $H^{-\varepsilon/2}$ respectively as $H\to 0$. Alternatively, at $H=0$, the correlation functions, or momentum-dependent susceptiblities diverge as p^{-2} and $p^{-\varepsilon}$ for small momenta p (see e.g. Schäfer and Horner 1978, Lawrie 1981 and references therein). More generally, one may expect singularities of this type to appear in all quantities associated with the response to the ordering field.

Unfortunately, this striking behaviour is not to be expected in real magnetic systems, where one never has perfect isotropy. In superfluids, on the other hand, the complex condensate wavefunction does provide a perfectly isotropic, two-component order parameter, but no ordering field is available in the laboratory. Here, one is restricted to studying the response to non-ordering perturbations, as revealed for example by specific heats and energy or entropy correlation functions (see e.g. Dohm and Folk 1980, 1981). From explicit calculations, it appears that the specific heat has a finite limit at the coexistence curve (Brézin *et al* 1974, Bervillier 1976, Chang and Houghton 1980). However, the general structure of non-ordering susceptibilities with respect to Goldstone modes does not seem to be widely understood, and it is this structure that we discuss in the present work.

In § 2, we describe how renormalisation group methods (Lawrie 1981) can be used to study Goldstone mode singularities in perturbation theory. The limiting form at coexistence of the generating function for non-ordering correlation functions is derived in § 3, and an exponentiated form of the entropy correlation function in the critical region is obtained in § 4. Our main conclusions are summarised in § 5.

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2. Perturbation theory

We work with the familiar O(N)-symmetric Landau-Ginzburg-Wilson Hamiltonian density

$$\mathscr{H} = \tfrac{1}{2}|\nabla \boldsymbol{\phi}|^2 + \tfrac{1}{2} r_0 \phi^2 + \frac{u_0}{4!}(\phi^2)^2 \tag{2.1}$$

in which $\boldsymbol{\phi}$ is an N-component field and $\phi^2 = \boldsymbol{\phi}\cdot\boldsymbol{\phi}$. Near the critical point, it suffices to take u_0 as a positive constant and r_0 as linear in temperature. Consequently, the singular part of the entropy correlation function is given by

$$C_0(x-y) = \langle \phi^2(x)\phi^2(y)\rangle_c \tag{2.2}$$

where the connected functions are

$$\langle AB\rangle_c = \langle (A - \langle A\rangle)(B - \langle B\rangle)\rangle \tag{2.3}$$

and the singular part of the specific heat is

$$C_0 = \int d^d x\, C_0(x). \tag{2.4}$$

Below the critical temperature, and in the absence of an ordering field, it is convenient to subtract from $\boldsymbol{\phi}$ its expectation value, and we write

$$\boldsymbol{\phi} = (\sigma + (3/u_0)^{1/2} m_0, \boldsymbol{\pi}) \tag{2.5}$$

where $m_0(r_0, u_0)$ is defined so that both the longitudinal field σ and the $(N-1)$-component transverse field $\boldsymbol{\pi}$ have zero expectation value. We then have

$$\begin{aligned} C_0(x-y) = {} & \langle \pi^2(x)\pi^2(y)\rangle_c + \langle \sigma^2(x)\sigma^2(y)\rangle_c + 2\langle \sigma^2(x)\pi^2(y)\rangle_c \\ & + 4(3/u_0)^{1/2} m_0(\langle \pi^2(x)\sigma(y)\rangle_c + \langle \sigma^2(x)\sigma(y)\rangle_c) + (12/u_0) m_0^2 \langle \sigma(x)\sigma(y)\rangle \end{aligned} \tag{2.6}$$

where $\pi^2(x) = \boldsymbol{\pi}(x)\cdot\boldsymbol{\pi}(x)$. The Feynman diagrams which contribute to this function up to one loop order are shown in figure 1. Three of these, namely (a), (c) and (i)

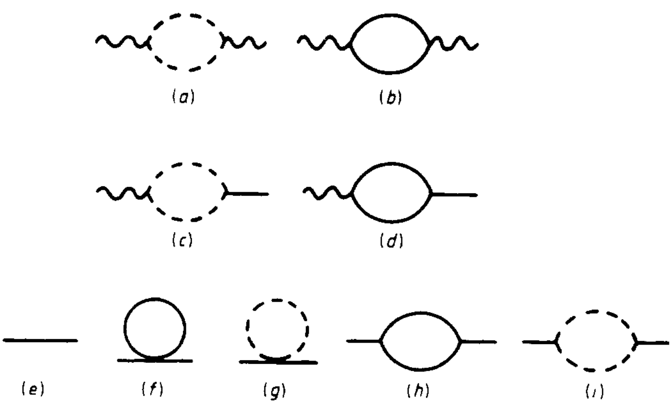

Figure 1. Tree and one-loop diagrams contributing to the entropy correlation function $C_0(x-y)$. Full and broken lines are respectively longitudinal (σ) and transverse (π) propagators. Wavy external lines correspond to the operator σ^2 or π^2 according to the internal lines to which they attach.

have infrared Goldstone singularities, owing to the massless transverse propagators. However, their sum is exactly zero, so that the function has no overall singularity at this order.

The question arises whether such cancellations are to be expected at all orders of perturbation theory and in other correlation functions. On the basis of an argument due originally to S Elitzur (see e.g. Amit and Kotliar (1980) and David (1981) for a discussion in the context of nonlinear σ models) one expects that any function constructed from $O(N)$ invariant operators such as ϕ^2 should indeed be infrared finite except possibly at the critical point. In fact, Chang and Houghton (1980) have obtained a finite two-loop expression for the specific heat, and one may check at low orders that, for example, the Goldstone mode contributions to $\langle\phi^2(x)\phi^2(y)\phi^2(z)\rangle_c$ do cancel, leaving a finite result.

For the two-dimensional $(\phi^2)^2$ model, Jevicki (1977) has noted that similar cancellations occur in low-order perturbative calculations of the ground-state energy, and has conjectured that they should persist to all orders. In two dimensions, of course, the spontaneous ordering which appears at the classical level is destroyed by Goldstone mode fluctuations, and there is thus no coexistence curve. It is therefore somewhat remarkable that some information can be obtained by perturbing about the classical ground state. (I thank a referee for reminding me of Jevicki's paper.)

The following sections discuss how the expectation of infrared finiteness is realised within a renormalisation group scheme which deals explicitly with Goldstone mode singularities (Lawrie 1981). In this scheme, a fixed point controlling behaviour on the coexistence curve is identified as follows.

(i) The longitudinal field σ is rescaled according to

$$\sigma = s/m \tag{2.7}$$

where m is a renormalised version of the order parameter m_0.

(ii) Each renormalised correlation function $F(\{p\}, u, m, \kappa)$ satisfies a relation of the form

$$F(\{\lambda p\}, u, m, \kappa) = m^{n_F}\kappa^{d_F}P_F(\lambda, u, m)f(\{p/\kappa\}, \bar{u}(\lambda), \bar{m}(\lambda)). \tag{2.8}$$

In this equation $\{p\}$ denotes a set of momentum arguments, u is a renormalised coupling constant and κ is an arbitrary renormalisation mass scale, which serves to make u and m dimensionless. The power n_F is that implied for the function F by (2.7) and d_F denotes the canonical dimension of F. The renormalisation group functions satisfy $P_F(1, u, m) = 1$, $\bar{u}(1) = u$ and $\bar{m}(1) = m$. In the infrared limit $\lambda \to 0$ with $m > 0$, the effective coupling $\bar{u}(\lambda)$ approaches a fixed point $u^{**} \sim \varepsilon/(N-1) + O(\varepsilon^2)$ while the effective mass diverges as $\bar{m}(\lambda) \simeq m/\lambda^{1-\varepsilon/2}$. The renormalisation prescription which leads to this behaviour incorporates the requirement that at each order of the double expansion in u and ε, $f(\{p\}, u, \infty)$ has a finite non-zero value. Consequently, the leading infrared singularity of F is contained in the prefactor $P_F(\lambda)$.

(iii) The leading singularity can be studied in the unrenormalised theory by making the substitution $\sigma = s/m_0$ and taking the limit $m_0 \to \infty$ at fixed u_0. This leads to a Gaussian fixed-point ensemble, which generates the functions $f(\{p\}, u, \infty)$ in terms of zero- and one-loop diagrams.

The leading singularity in (2.8) is contained in the asymptotic form of the prefactor $P_F^{**}(\lambda) = P_F(\lambda, u^{**}, \infty)$. One may write

$$f(\{\lambda p\}, u^{**}, \infty) = P_F^{**}(\lambda)f(\{p\}, u^{**}, \infty) \tag{2.9}$$

though this is a trivial statement, since $f(\{p\}, u^{**}, \infty)$ is exactly calculable. The singular behaviour of F may thus be expressed as

$$F(\{\lambda p\}, u, m, \kappa) \simeq m^{n_F} \kappa^{d_F} \tilde{P}_F(u, m) F(\{\lambda p\}, u^{**}, \infty) \tag{2.10}$$

where the amplitude $\tilde{P}_F(u, m)$ is finite on the coexistence curve but will, in general, exhibit a critical singularity as $m \to 0$. Near the critical point, (2.10) is valid only when the limit $\lambda \to 0$ is taken in advance of the limit $m \to 0$. More generally, one may expect that (2.10) will be a good approximation when the set of quantities $\{p/\kappa m^{\nu/\beta}\}$ are all small, where ν and β are the usual correlation length and order parameter exponents. We note that our description of the infrared singularities applies to the case in which all momenta $\{p\}$ assume uniformly small values. Cases of 'exceptional' momenta, in which some partial sums of the p vanish while others remain non-zero will, in general, produce infrared singularities not covered by the present discussion. Finally, if correlation functions are evaluated at zero momentum, but in the presence of an ordering field H, then the same Gaussian ensemble describes the limit $H \to 0$ or, indeed, the general case of small H and $\{p\}$. In these cases, m still denotes the zero-field value of the order parameter and σ has an expectation value roughly proportional to H.

3. Singularities on the coexistence curve

We now construct a generating functional for correlation functions constructed from the operators $\sigma(x)$ and $\phi^2(x) = \pi^2(x) + \sigma^2(x) + 2(3/u_0)^{1/2} m_0 \sigma(x) + 3m_0^2/u_0$. To this end, we augment the Hamiltonian (2.1) by writing

$$\mathcal{H}_s = \mathcal{H} - l_0(x)(\phi^2(x) - \langle\phi^2\rangle) - H_0(x)\sigma(x) - \tfrac{1}{2} l_0^2(x) B_0(u_0, r_0, \kappa) \tag{3.1}$$

where $\langle \ldots \rangle$ denotes an expectation value in the absence of the sources l_0 and H_0, and B_0 provides an additive renormalisation at mass scale κ of the entropy correlation function $\langle\phi^2(x)\phi^2(y)\rangle$. When analysing the critical singularities, B_0 can be taken as independent of r_0. For our purposes a dependence on r_0 is essential. Provided that this dependence is non-singular near the critical point $r_0 = r_{0c}$, the analysis of infrared behaviour is not prejudiced. According to § 2, the asymptotic behaviour involves the field $s = m_0\sigma$, and we therefore rescale the source H_0 by $H_0(x) = (3/u_0)^{1/2} m_0 h(x)$. The generating functional of connected functions is now defined by

$$\exp G(\{l_0\}, \{h\}) = \left[\int \mathcal{D}\phi \exp\left(-\int \mathrm{d}^d x\, \mathcal{H}_s(x)\right)\right]\left[\int \mathcal{D}\phi \exp\left(-\int \mathrm{d}^d x\, \mathcal{H}(x)\right)\right]^{-1}. \tag{3.2}$$

On taking the limit $m_0 \to \infty$, we find that $\mathcal{H}_s$ acquires the form

$$\mathcal{H}_s = \tfrac{1}{2}|\nabla \boldsymbol{\pi}|^2 + \tfrac{1}{2}h\pi^2 + \tfrac{1}{2}\bar{s}^2 - (6/u_0)(l_0 + \tfrac{1}{2}h)^2 + (6/u_0)(l_0 + \tfrac{1}{2}h)A + l_0\langle\pi^2\rangle - \tfrac{1}{2}l_0^2 B_0 \tag{3.3}$$

where the shifted longitudinal field

$$\bar{s} = s + \tfrac{1}{2}(u_0/3)^{1/2}\pi^2 + (3/u_0)^{1/2}A - 2(3/u_0)^{1/2}(l_0 + \tfrac{1}{2}h) \tag{3.4}$$

is now decoupled from the remainder of the Hamiltonian. The counterterm A is determined perturbatively by

$$r_0 = -\tfrac{1}{2}m_0^2 + A(u_0, m_0) \tag{3.5}$$

together with the condition $\langle\sigma\rangle = 0$. In the limiting ensemble (3.3), it has only the one-loop contribution

$$A = -\left(\frac{N-1}{6}\right) u_0 \int \frac{d^d q}{q^2} \tag{3.6}$$

required to satisfy $\delta G/\delta h = 0$ when $h = 0$, and we also have $\langle\pi^2\rangle = -6A/u_0$.

Since the limiting ensemble is Gaussian, the generating functional can be evaluated exactly. We obtain

$$G(\{l_0\}, \{h\}) = \int d^d x[(6/u_0)(l_0 + \tfrac{1}{2}h)^2 + \tfrac{1}{2}l_0^2 B_0 - (3/u_0)hA] - \tfrac{1}{2}(N-1)\,\mathrm{Tr}\ln(-\nabla^2 + h). \tag{3.7}$$

When l_0 and h are taken to be independent of position, we obtain the generating function for correlation functions at zero momentum:

$$V^{-1}G(l_0, h) = \left(\frac{6}{u_0}\right)\left[(l_0 + \tfrac{1}{2}h)^2 + (u_0/12)l_0^2 B_0 + \left(\frac{N-1}{12\varepsilon}\right) S u_0 h^{2-\varepsilon/2} B(1 + \tfrac{1}{2}\varepsilon, 1 - \tfrac{1}{2}\varepsilon)/(2 - \tfrac{1}{2}\varepsilon)\right] \tag{3.8}$$

where V is the volume of the system, $S = 2\pi^{d/2}/(2\pi)^d(\tfrac{1}{2}d - 1)!$ and $B(\alpha, \beta)$ is the Euler beta function. The one-loop term has been evaluated by standard dimensional regularisation methods and has a simple pole at $\varepsilon = 0$. This pole may be eliminated by introducing renormalised parameters u and l, defined by

$$u_0 = \kappa^\varepsilon u Z(u) \qquad l_0 = l Z(u) \tag{3.9}$$

where

$$Z(u) = [1 - (N-1)Su/6\varepsilon]^{-1} \tag{3.10}$$

and κ is an arbitrary mass scale, and by taking

$$B_0 = -2(N-1)(S/\varepsilon)\kappa^{-\varepsilon} Z^{-1}(u). \tag{3.11}$$

This gives

$$V^{-1}G(l, h) = (6/u)\kappa^{-\varepsilon}\left\{(l + \tfrac{1}{2}h)^2 + \left(\frac{N-1}{12\varepsilon}\right) S u h^2 \left[\frac{B(1 + \tfrac{1}{2}\varepsilon, 1 - \tfrac{1}{2}\varepsilon)}{2 - \tfrac{1}{2}\varepsilon}\left(\frac{\kappa^2}{h}\right)^{\varepsilon/2} - \frac{1}{2}\right]\right\}. \tag{3.12}$$

Of course, the same prescription (3.9)-(3.11) will serve to renormalise (3.7). Taking into account that the source l has canonical dimension two we obtain the renormalisation group functions

$$W(u) \equiv \kappa \left.\frac{\partial u}{\partial \kappa}\right|_{u_0} = -\varepsilon u[1 - (N-1)Su/6\varepsilon] \tag{3.13}$$

$$\gamma(u) \equiv 2 - \kappa \left.\frac{\partial \ln l}{\partial \kappa}\right|_{u_0, l_0} = 2 - (N-1)Su/6 \tag{3.14}$$

and identify the infrared-stable fixed point $Su^{**}=6\varepsilon/(N-1)$ and the scaling exponent $\gamma(u^{**})=2-\varepsilon$.

From (3.7) and (3.12) we see that, in the limiting coexistence ensemble, all the connected correlation functions involving $\phi^2(x)$ vanish except for $\langle\phi^2(x)\phi^2(y)\rangle_c$ and $\langle\phi^2(x)s(y)\rangle_c$ which are simply proportional to $\delta(x-y)$. Their Fourier transforms are therefore just finite constants. By contrast, all correlations of the longitudinal field, coupled to h, have infrared singularities arising from the one-loop integral. This result is unfortunately less powerful than it might appear, though it could be indicative of a stronger one. Let us ignore temporarily the explicit form of the generating functional and define the limiting correlation functions $g_0^{(n)}(x_1\ldots x_n)$ by

$$G(\{l_0\},0)=\sum_n\frac{1}{n!}\int \mathrm{d}^dx_1\ldots\mathrm{d}^dx_n\,l_0(x_1)\ldots l_0(x_n)g_0^{(n)}(x_1\ldots x_n). \tag{3.15}$$

The renormalisation induced by (3.9) is $g^{(n)}=Z^ng_0^{(n)}$ and, for the Fourier-transformed functions, dimensional analysis gives

$$\left(\kappa\frac{\partial}{\partial\kappa}+\lambda\frac{\partial}{\partial\lambda}+2n-d\right)g^{(n)}(\{\lambda p\},u,\kappa)=0. \tag{3.16}$$

Taking into account the explicit κ dependence of $g_0^{(2)}$ through B_0, we obtain the renormalisation group equation

$$\left(\lambda\frac{\partial}{\partial\lambda}-W\frac{\partial}{\partial u}+n\gamma-d\right)g^{(n)}(\{\lambda p\},u,\kappa\}=-B\delta_{n,2} \tag{3.17}$$

where

$$B\equiv Z^2\kappa\frac{\partial B_0}{\partial\kappa}\bigg|_{u_0}=2(N-1)\kappa^{-\varepsilon}S. \tag{3.18}$$

At the fixed point $u=u^{**}$, the solution is

$$g^{(n)}(\{\lambda p\},u^{**},\kappa)=\lambda^{d-(2-\varepsilon)n}\left[g^{(n)}(\{p\},u^{**},\kappa)-\left(\frac{12}{\kappa^{\varepsilon}u^{**}}\right)\delta_{n,2}\right]+\left(\frac{12}{\kappa^{\varepsilon}u^{**}}\right)\delta_{n,2} \tag{3.19}$$

which, apart from the inhomogeneous term for $n=2$ is of the form (2.9). Now (3.12) shows that this is trivially satisfied by

$$g^{(n)}(\{p\},u,\kappa)=\left(\frac{12}{\kappa^{\varepsilon}u}\right)\delta_{n,2} \tag{3.20}$$

and that the amplitudes of the leading singularities in each of the original correlation functions vanish identically. This is clearly a consequence of symmetry and assures us that the perturbation theory cancellations noted in § 2 do indeed persist to all orders and in all correlation functions constructed from the invariant operator ϕ^2. However, this does not prove, in general, that the original functions $G^{(n)}(\{p\},u,m,\kappa)$, whose limits are

$$G^{(n)}(\{p\},u^{**},\infty,\kappa)=g^{(n)}(\{p\},u^{**},\kappa) \tag{3.21}$$

vanish, or are even finite in the infrared limit, though we suspect that some such result may hold. This is because, for $n>2$, the leading power of λ in (3.19) is negative, and the possibility that corrections, proportional to powers of $(\lambda^{2-\varepsilon}/m^2)$ for finite m could yield residual, though less severe singularities is not ruled out by our arguments. The analysis of such corrections, which are not merely integer powers of $(\lambda^{2-\varepsilon}/m^2)$, does not seem to be at all straightforward.

In the case $n=2$, which is of the greatest physical interest since it gives the singular part of the entropy correlation function and specific heat, the leading power of λ in (3.19) is positive (below four dimensions). The preceding remarks do not apply and (3.20) can be taken at face value. However, (3.20) does not yield the critical singularity, which is contained in a prefactor similar to $\tilde{P}_F$ in (2.10) and must vary as $m^{-\alpha/\beta}$ as $m\to 0$. This is studied in more detail in the following section.

4. Specific heat and entropy correlation function

In this section, we use the full renormalised theory, to order ε, to exhibit both the crossover to critical behaviour and the general momentum dependence of the entropy correlation function. It is convenient to expand the generating functional (3.2) as

$$G=\sum_{m,n}\frac{1}{m!\,n!}G_0^{(m,n)}(H_0+2Ml_0)^m l_0^n \tag{4.1}$$

where $M=|\langle\boldsymbol{\phi}\rangle|=(3/u_0)^{1/2}m_0$ and integrations over coordinate arguments as in (3.15) are implied. In this representation, l_0 couples to $(\pi^2+\sigma^2)$ and (H_0+2Ml_0) couples to σ. The functions $G_0^{(m,n)}$ are the connected correlation functions of the corresponding operators except that $G_0^{(0,2)}$ also contains the counterterm B_0. One-particle irreducible vertex functions may as usual be obtained by Legendre transformation. The first few are given by

$$\Gamma_0^{(2,0)}=1/G_0^{(2,0)} \tag{4.2}$$

$$\Gamma_0^{(1,1)}=2M+G_0^{(1,1)}/G_0^{(2,0)} \tag{4.3}$$

$$\Gamma_0^{(0,2)}=G_0^{(0,2)}-(G_0^{(1,1)})^2/G_0^{(2,0)}. \tag{4.4}$$

The entropy correlation function, defined as

$$C_0(x-y)=\langle\phi^2(x)\phi^2(y)\rangle_c=\left[\frac{\delta^2 G}{\delta l_0(x)\,\delta l_0(y)}\bigg|_{H_0}\right]_{H_0=l_0=0} \tag{4.5}$$

may be expressed as

$$C_0=\Gamma_0^{(0,2)}+(\Gamma_0^{(1,1)})^2/\Gamma_0^{(2,0)}. \tag{4.6}$$

The functions $G_0^{(m,n)}$ are not multiplicatively renormalisable; their utility lies in the ease of identifying Feynman diagrams which contribute to them. However, the vertex functions $\Gamma_0^{(m,n)}$ are multiplicatively renormalisable. We have described in detail elsewhere (Lawrie 1981) a renormalisation scheme which interpolates between the coexistence limit of § 3 and the critical limit. Briefly, we define perturbatively a renormalised temperature-like parameter τ which, when it is small is proportional to $(r_{0c}-r_0)$ and when it is large is proportional to m_0^2. In a region near the critical point, including the coexistence curve, τ is always small, and may be taken as linear in (T_c-T). However, the running parameter $\bar{\tau}(\lambda)$ into which it is mapped by the renormalisation group becomes infinite at the coexistence curve, and the renormalisation scheme ensures that the vertex functions remain finite in this limit after extraction of an overall power of τ corresponding to m^{n_F} in (2.8).

At one-loop order, we find that the renormalised vertex functions $\Gamma^{(m,n)} = Z_{\phi^2}^{-n}\Gamma_0^{(m,n)}$ satisfy the renormalisation group equation

$$\left(\lambda\frac{\partial}{\partial\lambda} - W\frac{\partial}{\partial u} + \gamma\tau\frac{\partial}{\partial\tau} - d + (\tfrac{1}{2}d-1)m + \gamma n\right)\Gamma^{(m,n)}(\{\lambda p\}, u, \tau, \kappa)$$
$$= -\delta_{m,0}\,\delta_{n,2}\,B \tag{4.7}$$

where

$$W = -\varepsilon u + \left(\frac{N+8}{6} - \frac{3\tau}{3+2\tau}\right)Su^2 \tag{4.8}$$

$$\gamma = 2 - \left(\frac{N+2}{6} - \frac{\tau}{3+2\tau}\right)Su \tag{4.9}$$

$$B = 2\left(N - \frac{2\tau}{3+2\tau}\right)\kappa^{-\varepsilon}S. \tag{4.10}$$

We have ignored the wavefunction renormalisation which is required only at higher orders. In the limit $\tau\to\infty$, we evidently reproduce (3.13), (3.14), (3.17) and (3.18). At the critical point $\tau=0$ we have the usual fixed point $Su^* = 6\varepsilon/(N+8) + O(\varepsilon^2)$ and $\nu = 1/\gamma(u^*) = \frac{1}{2}[1 + (N+2)\varepsilon/2(N+8) + O(\varepsilon^2)]$ reproduces the usual expansion for the correlation length exponent.

It is helpful, as suggested by (3.12), to define rescaled vertex functions $\hat{\Gamma}^{(m,n)} = (u/12)^{n/2}\Gamma^{(m,n)}$, which satisfy the equation

$$\left(\lambda\frac{\partial}{\partial\lambda} - W\frac{\partial}{\partial u} + \gamma\tau\frac{\partial}{\partial\tau} - d + (\tfrac{1}{2}d-1)m + (\tfrac{1}{2}d + \hat{\alpha})n\right)\hat{\Gamma}^{(m,n)}(\{\lambda p\}, u, \tau, \kappa)$$
$$= -\delta_{m,0}\,\delta_{n,2}(u/12)B \tag{4.11}$$

where

$$\hat{\alpha}(u,\tau) = \gamma - W/2u - \tfrac{1}{2}d = \frac{1}{2}\left(\frac{4-N}{6} - \frac{\tau}{3+2\tau}\right)Su. \tag{4.12}$$

At the coexistence fixed point, we have $\hat{\alpha}(u^{**},\infty) = -\frac{1}{2}\varepsilon$, while at the critical point, the expression

$$\hat{\alpha}(u^*, 0) = (4-N)\varepsilon/2(N+8) + O(\varepsilon^2) \tag{4.13}$$

may be identified as the combination $\alpha/2\nu$ of critical exponents.

To evaluate the renormalised correlation function $C = Z_{\phi^2}^{-2}C_0$, we need vertex functions for which $m+n=2$. For these, the solution of (4.11) may be written as

$$\hat{\Gamma}^{(2-n,n)}(p,u,\tau) = \lambda^{2-n}P^{n/2}(\lambda,u,\tau)\hat{\Gamma}^{(2-n,n)}(p/\lambda, \bar{u}(\lambda), \bar{\tau}(\lambda)) + \delta_{n,2}R(\lambda,u,\tau) \tag{4.14}$$

where, as usual, λ is an arbitrary scale parameter, and characteristic functions are defined as the solutions of

$$\lambda\frac{\partial\bar{u}}{\partial\lambda} = W(\bar{u},\bar{\tau}) \tag{4.15}$$

$$\lambda\frac{\partial\bar{\tau}}{\partial\lambda} = -\gamma(\bar{u},\bar{\tau})\bar{\tau} \tag{4.16}$$

$$\lambda\frac{\partial P}{\partial\lambda}=-2\hat{\alpha}(\bar{u},\bar{\tau})P \tag{4.17}$$

$$\lambda\frac{\partial R}{\partial\lambda}=-P(\lambda)(\bar{u}/12)B(\bar{u},\bar{\tau}) \tag{4.18}$$

with the initial conditions $\bar{u}=u$, $\bar{\tau}=\tau$, $P=1$, $R=0$ at $\lambda=1$. The functions $\hat{\Gamma}^{(2-n,n)}$ are given, to first order of the expansion in u and ε by

$$\hat{\Gamma}^{(0,2)}=\left(\frac{N-1}{12}\right)Su(1-\ln p^2)-\left(\frac{Su}{12}\right)(1+I(p^2,\tau)) \tag{4.19}$$

$$\hat{\Gamma}^{(1,1)}=(2\tau)^{1/2}\left[1-\tfrac{1}{4}Su\ln\left(\frac{2\tau}{3+2\tau}\right)+\left(\frac{N-1}{12}\right)Su(\ln p^2-1)+\left(\frac{Su}{4}\right)(1+I(p^2,\tau))\right] \tag{4.20}$$

$$\hat{\Gamma}^{(2,0)}=p^2+2\tau\left[1-\tfrac{1}{2}Su\ln\left(\frac{2\tau}{3+2\tau}\right)+\left(\frac{N-1}{12}\right)Su(\ln p^2-1)+\tfrac{3}{4}Su(1+I(p^2,\tau))\right] \tag{4.21}$$

where

$$I(p^2,\tau)=\int_0^1 \mathrm{d}x\ln\left(\frac{p^2x(1-x)+2\tau}{3+2\tau}\right) \tag{4.22}$$

and the mass scale κ, which plays no further role has been set equal to one. Each of these functions has a logarithmic singularity when $p^2\to 0$, but remains finite when $\tau\to\infty$. On the right-hand side of (4.14), we therefore exponentiate the singularities by choosing $\lambda=|p|$.

We offer approximate solutions to (4.15)-(4.18) which have all the required analytic properties in $\lambda=|p|$ and τ. We define first the auxiliary function Q by

$$Q^{-1}=1+\left(\frac{N-1}{N+8}\right)^{2/(2-\varepsilon)}\left[\left(\frac{3+2\bar{\tau}}{3+2\tau}\right)^{\varepsilon/(2-\varepsilon)}-1\right]. \tag{4.23}$$

Then $\bar{\tau}$ is defined implicitly by

$$\bar{\tau}=\tau|p|^{-1/\nu}\left(\frac{3+2\bar{\tau}}{3+2\tau}\right)^{1-1/2\nu}Q^{1/2\nu} \tag{4.24}$$

and $\bar{u}$ by

$$\bar{u}=u^*\left(\frac{3+2\bar{\tau}}{3+2\tau}\right)^{\varepsilon/2}Q^{1-\varepsilon/2}. \tag{4.25}$$

Finally, we give

$$P=|p|^{-\alpha/\nu}\left(\frac{3+2\bar{\tau}}{3+2\tau}\right)^{-\alpha/2\nu}Q^{1+\alpha/2\nu} \tag{4.26}$$

$$R=1+\left(\frac{4}{4-N}\right)(\tau^{-\alpha}-1)-\left(\frac{\tau^{-\alpha}}{4-N}\right)\times\left[4-N(|p|\tau^{-\nu})^{-\alpha/\nu}\left(\frac{3+2\bar{\tau}}{3+2\tau}\right)^{-\alpha/2\nu}Q^{\alpha/2\nu}\right]Q. \tag{4.27}$$

One may verify that these are indeed solutions, up to corrections of order ε^2. Although they differ from the solutions given by Lawrie (1981), they are equivalent to these at order ε. The rationale for the particular expressions we have written down lies in their analytic structure, and limiting behaviour which is implied by the renormalisation group equations, and which we now discuss.

(i) The limit $|p|=1$. As required by the initial conditions, we have $Q=P=1$, $\bar{u}=u^*$, $\bar{\tau}=\tau$, $R=0$. The special choice $u=u^*$, which eliminates some corrections to scaling, has been made to facilitate the solutions.

(ii) The limit $\tau\to 0$, $|p|\to 0$. We keep the ratio $q=|p|\tau^{-\nu}$ fixed. When q is large, we have the singularity characteristic of the critical point, namely $\bar{\tau}\simeq q^{-1/\nu}$, $Q\simeq 1$, $\bar{u}\simeq u^*$, $P\simeq |p|^{-\alpha/\nu}$, $R\simeq N|p|^{-\alpha/\nu}/(4-N)$, with corrections appearing as integer powers of $q^{-1/\nu}$.

(iii) The limit $|p|\to 0$, $\tau>0$, $N\neq 1$. In this limit q is small, and we have the singularity characteristic of the coexistence curve. In the critical region, we may take $|p|^{1/\nu}\ll\tau\ll\frac{3}{2}$. Then $\bar{\tau}$ is large, and we have

$$Q\simeq\left(\frac{N+8}{N-1}\right)^{2/(2-\varepsilon)}(\tfrac{2}{3}\bar{\tau})^{-\varepsilon/(2-\varepsilon)} \tag{4.28}$$

$$\bar{u}\simeq u^{**}=u^*\left(\frac{N+8}{N-1}\right)+\mathrm{O}(\varepsilon^2). \tag{4.29}$$

If we define the function $t(q)$ by $\bar{\tau}=q^{\varepsilon-2}t(q)$, then (4.24) gives

$$t=(q^{2-\varepsilon}+\tfrac{2}{3}t)^{1-1/2\nu}\left[q^{\varepsilon}+\left(\frac{N-1}{N+8}\right)^{2/(2-\varepsilon)}[(q^{2-\varepsilon}+\tfrac{2}{3}t)^{\varepsilon/(2-\varepsilon)}-q^{\varepsilon}]\right]^{-1/2\nu} \tag{4.30}$$

with

$$\tfrac{2}{3}t(0)=(\tfrac{2}{3})^{(2-\varepsilon)\nu}\left(\frac{N+8}{N-1}\right). \tag{4.31}$$

We see that $t(q)$ has a double expansion in q^{ε} and $q^{2-\varepsilon}$ which will, of course, contain integer powers of q^2. Corrections to these exponents of order ε^2 are not expected, since the limiting coexistence ensemble is Gaussian. For P and R we have

$$P=\tau^{-\alpha}\left(\frac{N+8}{N-1}\right)(\tfrac{2}{3})^{-(\alpha+\varepsilon\nu)}q^{\varepsilon}+\dots \tag{4.32}$$

$$R=1+\frac{4}{4-N}(\tau^{-\alpha}-1)-\frac{\tau^{-\alpha}}{4-N}\left(\frac{N+8}{N-1}\right)(\tfrac{2}{3})^{-\varepsilon\nu}[4-(\tfrac{2}{3})^{-\alpha}N]q^{\varepsilon}+\dots \tag{4.33}$$

where the dots denote higher powers of q^{ε} and $q^{2-\varepsilon}$.

(iv) The limit $|p|\to 0$, $\tau>0$, $N=1$. When $N=1$, we have $Q=1$. The limiting behaviour of $\bar{\tau}$ may be described by $\bar{\tau}=q^{-2}t(q)$ where $t(q)$ satisfies

$$t^{2\nu}=(\tfrac{2}{3}t+q^2)^{2\nu-1}\qquad t(0)=(\tfrac{2}{3})^{2\nu} \tag{4.34}$$

and is clearly analytic in q^2. This property, expected for an Ising-like system away from its critical point, carries over to the functions P and R. Sadly, we also have $\bar{u}\simeq u^*q^{-\varepsilon}$, so that when one substitutes the expressions (4.19)-(4.21) into the right-hand side of (4.14), some weak, artificial singularities remain. This is because the fixed point u^{**} no longer exists when $\bar{\tau}\to\infty$, and the choice $\lambda=|p|$ is inappropriate. Since the $\ln p^2$ terms are now absent, we may instead impose, say, the condition $\bar{\tau}=1$. Now $|p|$ in (4.24)-(4.27) is replaced by λ and the condition $\bar{\tau}=1$ implies $\lambda\simeq\tau^{\nu}$ and $p/\lambda\simeq q$. Each correlation function is now analytic in q^2 and Goldstone mode singularities are absent, as they must be.

If we ignore residual corrections from the one-loop terms in (4.19)-(4.21) the entropy correlation function is given by

$$\frac{u^*}{12}C(p,\tau)\simeq R+\frac{P}{1+(2\bar{\tau})^{-1}} \tag{4.35}$$

which describes the leading singularities as delivered by the renormalisation group. Some further ingenuity is needed to effect a complete exponentiation. For small q, this function may be written as

$$C(p,\tau)\simeq\left(\frac{12}{u^*}\right)+\left(\frac{4}{\alpha}\right)(\tau^{-\alpha}-1)+\left(\frac{N+8}{N-1}\right)\left(\frac{4}{\alpha}\right)\tau^{-\alpha}(\tfrac{3}{2})^{\varepsilon\nu}[(\tfrac{3}{2})^{\alpha}-1]q^{\varepsilon}+\dots. \tag{4.36}$$

The first term clearly corresponds to (3.20) with $n=2$, and the second contains the usual critical singularity of the specific heat. The third term represents the leading momentum-dependent correction, and we note that its amplitude remains finite and non-zero when the exponent α vanishes. Nicoll (1980) has also obtained an expression for $C(p,\tau)$ in which the corrections vanish as p^{ε}.

5. Conclusions

We have studied the effect of Goldstone modes on correlation functions involving the O(N) invariant operator $(\phi^2-\langle\phi^2\rangle)$ for an exactly isotropic system below its critical point. In the limiting ensemble which describes the zero-momentum behaviour of these functions, the source of this operator (that is, the field thermodynamically conjugate to it) does not couple to the transverse fields. Consequently, when the functions are evaluated in perturbation theory, the purely transverse diagrams are guaranteed to cancel exactly at each order. In particular, the entropy correlation function has a zero-momentum limit which yields a finite, non-zero specific heat. However, residual Goldstone mode singularities appear in the momentum-dependent part, which may be expressed as an infinite series of terms of the form $|p|^{m\varepsilon+n(2-\varepsilon)}$ where m and n are positive integers. In three dimensions, this amounts to a power series containing both even and odd powers of $|p|$. For Ising-like systems, $N=1$, one obtains only integer powers of p^2. We have obtained a one-loop approximation to the entropy correlation function which exhibits these features, as well as the usual critical scaling properties. However, the numerical details should probably not be taken too seriously.

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